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On the Stability of a Solution of a Guarantee Optimization Problem under a Functional Constraint on the Disturbance ⋆

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Abstract: The paper deals with a control problem for a dynamical system under disturbances. In addition to geometric constraints on the disturbance, it is supposed that all disturbance realizations belong to some unknown L_1-compact set. The control is aimed at minimization of a given quality index. Within the game-theoretical approach, the problem of optimizing the guaranteed result is studied. For solving this problem, we use a control procedure with a guide. The paper is focused on the questions of stability of this control procedure with respect to informational and computational errors. The results are illustrated by numerical simulation.

Keywords: control with a guide, disturbances, optimal guaranteed result, functional constraint.

1. INTRODUCTION

The paper deals with a control problem for a dynamical system under disturbances. A motion of the system is considered on a finite interval of time and is described by a nonlinear ordinary differential equation. The values of the control and disturbance are subject to geometric constraints. In addition, we suppose that the disturbance satisfies a compact functional constraint (see Kryazhimskii (1991) and also Serkov (2013, 2015)): all disturbance realizations belong to a set that is compact in the space L_1. The control is aimed to minimize a quality index. Within the game-theoretical approach, a problem of optimizing the guaranteed result of the control is considered. In Gomoyunov and Serkov (2017, 2019), a control procedure with a guide is proposed for numerical solution of the problem. In this paper, we study stability properties of this control procedure with respect to informational and computational errors. The results obtained are illustrated by numerical simulation of the control process.

2. STATEMENT OF THE PROBLEM

2.1 Dynamical System

We consider a dynamical system which motion is described by the following differential equation:

$$\begin{aligned} &\dot{x}(t) = f(t, x(t), u(t), v(t)),\ t \in T = [t_0, \vartheta], \\ &x(t) \in \mathbb{R}^n,\ u(t) \in P \subset \mathbb{R}^p,\ v(t) \in Q \subset \mathbb{R}^q. \end{aligned} \tag{1}$$

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Here t is the time, x is the state vector, $\dot{x}(t) = \mathrm{d}x(t)/\,\mathrm{d}t$; u is the control vector, v is the disturbance vector; t_0 and ϑ are the initial and terminal times; P and Q are known compact sets.

It is assumed that the function $f : T \times \mathbb{R}^n \times P \times Q \to \mathbb{R}^n$ satisfies the following assumptions: f is continuous; for any compact set $D \subset \mathbb{R}^n$, there exists $L > 0$ such that, for any $t \in T$, $x, x' \in D$, $u \in P$, $v \in Q$, the inequality $\|f(t, x, u, v) - f(t, x', u, v)\| \le L\|x - x'\|$ is valid; there exists $a > 0$ such that $\|f(t, x, u, v)\| \le a(1 + \|x\|)$ for any $t \in T$, $x \in \mathbb{R}^n$, $u \in P$, $v \in Q$. Here and below the symbol $\|\cdot\|$ denotes the Euclidian norm.

Note that these assumptions are quite standard for the differential games theory (see, e.g., (Krasovskii and Subbotin, 1988, §1.2) and also (Krasovskii and Krasovskii, 1995, §3)).

A pair $(t, x) \in T \times \mathbb{R}^n$ is called a position of system (1). Following (Krasovskii and Krasovskii, 1995, §3), we define the set G of all possible positions as follows:

$$G = \big\{(t, x) \in T \times \mathbb{R}^n : \|x\| \le (1+R_0)e^{(t-t_0)a} - 1\big\},\ R_0 > 0.$$

By admissible realizations $u(\cdot)$ of the control and $v(\cdot)$ of the disturbance, we mean (Lebesgue) measurable functions $u : T \to P$ and $v : T \to Q$. The sets of all such realizations are denoted by $\mathcal{U}$ and $\mathcal{V}$. One can show that, due to the properties of the function f, for any initial state

$$x_0 \in G_0 = \big\{x \in \mathbb{R}^n : \|x\| \le R_0\big\}$$

and any $u(\cdot) \in \mathcal{U}$, $v(\cdot) \in \mathcal{V}$, there exists a unique motion $x(\cdot) = x(\cdot; x_0, u(\cdot), v(\cdot))$ of system (1), which is an absolutely continuous function $x : T \to \mathbb{R}^n$ that satisfies the initial condition $x(t_0) = x_0$ and, together with $u(\cdot)$,

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$v(\cdot)$, satisfies equation (1) for almost every $t \in T$. Note that the inclusions $(t, x(t)) \in G$ are valid for $t \in T$.

2.2 Quality Index

Let quality of the control process be evaluated by the index

$$\gamma = \gamma(x(\cdot)) = \Big(\sum_{i=1}^{N} \| D_i(x(\vartheta_i) - c_i) \|^2 \Big)^{1/2}. \tag{2}$$

Here the times $\vartheta_i \in T$ are given such that $\vartheta_i < \vartheta_{i+1}$, $i \in 1..(N-1)$, $\vartheta_N = \vartheta$; $c_i \in \mathbb{R}^n$ are target vectors and D_i are constant $(n \times n)$-matrices, $i \in 1..N$.

The goal of the control is to minimize the value γ of quality index (2). Since there are unknown disturbances acting in system (1), in accordance with the guaranteed result principle, we take into account that, in the worst case, the disturbances may aim at maximization of γ.

2.3 Functional Constraint on the Disturbance

Let us suppose that, in addition to the geometric constraint $v(t) \in Q$, $t \in T$, the disturbance satisfies a functional constraint, imposed not on the instantaneous values $v(t)$, but on the realization $v(\cdot)$ as a whole. Let $L_1(T, \mathbb{R}^q)$ denote the space of all (classes of) summable functions from T to $\mathbb{R}^q$ with the standard norm. Following Kryazhimskii (1991); Serkov (2013), by a compact functional constraint on the disturbance, we mean a family $\mathbb{V} = \{V\}$ of compact in $L_1(T, \mathbb{R}^q)$ sets $V \subset \mathcal{V}$ such that $\mathcal{V} = \bigcup_{V \in \mathbb{V}} V$. We say that the disturbance satisfies the constraint $\mathbb{V}$ if there exists a set $V \in \mathbb{V}$ such that every disturbance realization $v(\cdot)$ that can happen in system (1) satisfies the inclusion $v(\cdot) \in V$. Thus, it is assumed that, when forming the control actions, we know only the constraint $\mathbb{V}$, but the specific set $V \in \mathbb{V}$ is not given.

This notion of a functional constraint is quite general and can be used in order to formalize an additional information about the disturbance. Let us give two typical examples:

- It is known that every realization $v(\cdot)$ is a piecewise constant function with a fixed number of possible discontinuity points, however, this number is unknown.
- It is known that every realization $v(\cdot)$ is a continuous function with a fixed modulus of continuity, however, this modulus is unknown.

The paper deals with a guarantee optimization problem for system (1) and quality index (2) in the case when the disturbance satisfies a compact functional constraint $\mathbb{V}$.

3. OPTIMAL GUARANTEED RESULT

Following Ryll-Nardzewski (1964), by a quasi-strategy, we mean a function $\alpha : \mathcal{V} \to \mathcal{U}$ with the property of non-anticipation: if, for any $t \in T$ and any $v(\cdot), v'(\cdot) \in \mathcal{V}$, the equality $v(\tau) = v'(\tau)$ is valid for almost every $\tau \in [t_0, t]$, then the corresponding images $u(\cdot) = \alpha(v(\cdot))$ and $u'(\cdot) = \alpha(v'(\cdot))$ satisfy the equality $u(\tau) = u'(\tau)$ for almost every $\tau \in [t_0, t]$. For any initial state $x_0 \in G_0$, we define the value of the optimal guaranteed result as follows:

$$\Gamma^0(x_0) = \inf_{\alpha} \sup_{v(\cdot) \in \mathcal{V}} \gamma\big(x(\cdot; x_0, \alpha(v(\cdot)), v(\cdot))\big).$$

From the results of (Krasovskii, 1985, §§28, 29) (see also (Krasovskii and Krasovskii, 1995, §9)) it follows that, under the considered conditions, the value $\Gamma^0(x_0)$ can be guaranteed by means of positional counter-strategies. Namely, there exists an optimal counter-strategy

$$U^0(t, x, v, \varepsilon) \in P, \ (t, x) \in G, \ v \in Q, \ \varepsilon > 0,$$

which, for any t, x and ε, is Borel measurable as a function of v, and such that the proposition below is valid.

Proposition 1. For any $\zeta > 0$, there exist $\varepsilon^0 > 0$ and a function $\delta^0(\varepsilon) > 0$, $\varepsilon \in (0, \varepsilon^0]$, such that, for any initial state $x_0 \in G_0$, any $\varepsilon \in (0, \varepsilon^0]$ and any partition

$$\Delta = \big\{ \tau_i : \tau_0 = t_0, \, \tau_{i-1} < \tau_i, \, i \in 1..n_\Delta, \, \tau_{n_\Delta} = \vartheta \big\} \tag{3}$$

of the segment $T = [t_0, \vartheta]$ such that $\max_{i \in 1..n_\Delta}(\tau_i - \tau_{i-1}) \leq \delta^0(\varepsilon)$, the following statement holds. For any disturbance realization $v(\cdot) \in \mathcal{V}$, the control law $\{U^0(\cdot), \varepsilon, \Delta\}$ that forms a control realization $u(\cdot) \in \mathcal{U}$ by the rule

$$u(t) = U^0(\tau_i, x(\tau_i), v(t), \varepsilon), \ t \in [\tau_i, \tau_{i+1}), \ i \in 0..(n_\Delta - 1),$$

provides the inequality

$$\gamma \leq \Gamma^0(x_0) + \zeta \tag{4}$$

for the corresponding value γ of quality index (2).

The practical use of counter-strategies is often complicated by the unavailability of direct measurements of current values $v(t)$ of the disturbance. On the other hand, in the general case, when the equilibrium condition in the small game (see, e.g., (Krasovskii and Krasovskii, 1995, p. 46)), or, in another terminology, the Isaac's condition (see Isaacs (1965)), is not assumed, the value $\Gamma^0(x_0)$ cannot be guaranteed if the control is formed without information on $v(t)$. However, as shown in Kryazhimskii (1991); Serkov (2013, 2015), the value $\Gamma^0(x_0)$ can be guaranteed without such information in the case when the disturbance satisfies a compact functional constraint $\mathbb{V}$. Namely, in Gomoyunov and Serkov (2019), a control procedure with a guide is proposed that guarantees the value $\Gamma^0(x_0)$. The procedure involves a mechanism of dynamical "reconstruction" of the disturbance together with the use of the optimal counter-strategy U^0. In Sect. 4, 5, we study the stability properties of this procedure with respect to informational and computational errors.

4. CONTROL PROCEDURE WITH A GUIDE

Let $x_0 \in G_0$, $\varepsilon \in (0, 1)$ and Δ be a partition (3). For simplicity, we suppose that the partition Δ has the constant step $\delta > 0$. Let us fix the parameters $\alpha > 0$ and $\beta > 0$ that determine the admissible levels of informational and computational errors.

For the compact set P, describing the geometric constraint on the control, we choose an ε–net $(u_j^\varepsilon)_{j \in 1..n_\varepsilon} \subset P$:

$$\max_{u \in P} \min_{j \in 1..n_\varepsilon} \|u - u_j^\varepsilon\| \leq \varepsilon, \tag{5}$$

and introduce extra times

$$\tau_i' = \tau_i - \varepsilon\delta, \ \tau_{ij}' = \tau_i' + j(\tau_i - \tau_i')/n_\varepsilon, \tag{6}$$

where $i \in 1..(n_\Delta - 1)$, $j \in 0..n_\varepsilon$.

We consider a control procedure that uses an auxiliary motion $y(\cdot)$ of system (1) as a guide (see, e.g., (Krasovskii and Subbotin, 1988, §8.2)). We suppose that, instead of x_0, the approximate value $\widetilde{x}_0$ is known such that

$$\widetilde{x}_0 \in G_0, \ \|x_0 - \widetilde{x}_0\| \leq \alpha. \tag{7}$$

We assume that the motion $y(\cdot)$ of the guide starts from the initial state $\widetilde{x}_0$ and denote by $\overline{u}(\cdot) \in \mathcal{U}$ and $\overline{v}(\cdot) \in \mathcal{V}$ the realizations of the control and "disturbance" generating this motion $y(\cdot)$. Thus, we have $y(\cdot) = x(\cdot; \widetilde{x}_0, \overline{u}(\cdot), \overline{v}(\cdot))$. Note that, for any such motion $y(\cdot)$, the inclusions $(t, y(t)) \in G$, $t \in T$, are valid. Let us describe a step-by-step procedure of forming a control realization $u(\cdot) \in \mathcal{U}$ in the original system and piecewise constant realizations $\overline{u}(\cdot)$ and $\overline{v}(\cdot)$ of the form

$$\overline{u}(t) = \overline{u}_i,\ \overline{v}(t) = \overline{v}_i,\ t \in [\tau_i, \tau_{i+1}),\ i \in 0..(n_\Delta - 1), \quad (8)$$

in the guide.

Let $i \in 0..(n_\Delta - 1)$. If $i = 0$, we choose $\overline{v}_0 \in Q$ arbitrarily. For $i > 0$, we assume that approximate values $\widetilde{x}(\tau'_{ij})$, $j \in 0..n_\varepsilon$, of the state vector are known such that

$$(\tau'_{ij}, \widetilde{x}(\tau'_{ij})) \in G,\ \|x(\tau'_{ij}) - \widetilde{x}(\tau'_{ij})\| \le \alpha, \quad (9)$$

and, as a "reconstruction" of the disturbance acting on the interval $[\tau_{i-1}, \tau_i)$, we choose $\overline{v}_i \in Q$ from the condition

$$\begin{aligned} &\max_{j \in 1..n_\varepsilon} \|\widetilde{d}_{ij} - f(\tau_i, \widetilde{x}(\tau_i), u^\varepsilon_j, \overline{v}_i)\| \\ &\le \min_{v \in Q} \max_{j \in 1..n_\varepsilon} \|\widetilde{d}_{ij} - f(\tau_i, \widetilde{x}(\tau_i), u^\varepsilon_j, v)\| + \beta, \end{aligned} \quad (10)$$

where $\widetilde{x}(\tau_i) = \widetilde{x}(\tau'_{in_\varepsilon})$ according to (9) and

$$\widetilde{d}_{ij} = (\widetilde{x}(\tau'_{ij}) - \widetilde{x}(\tau'_{i(j-1)}))/(\tau'_{ij} - \tau'_{i(j-1)}). \quad (11)$$

Using the optimal counter–strategy in the guide, we define

$$\overline{u}_i = U^0(\tau_i, y(\tau_i), \overline{v}_i, \varepsilon), \quad (12)$$

and, after that, we set

$$u(t) = \begin{cases} \overline{u}_i, & \text{if } t \in [\tau_i, \tau'_{i+1}), \\ u^\varepsilon_j, & \text{if } t \in [\tau'_{(i+1)(j-1)}, \tau'_{(i+1)j}),\ j \in 1..n_\varepsilon. \end{cases} \quad (13)$$

Here u^ε_j are the elements of the chosen ε–net.

Theorem 1. Let $\mathbb{V}$ be a compact functional constraint on the disturbance. Then, for any $\zeta > 0$, there exists $\varepsilon^* \in (0, 1)$ such that, for any $\varepsilon \in (0, \varepsilon^*]$ and any $V \in \mathbb{V}$, one can choose $\delta^* > 0$ such that, for any initial state $x_0 \in G_0$ and any partition Δ (3) with the constant step $\delta \le \delta^*$, one can specify the values of the parameters $\alpha > 0$ and $\beta > 0$ such that the control procedure with a guide (5)–(13) provides inequality (4) for any $v(\cdot) \in V$.

The proof of the theorem is carried out by the scheme from (Gomoyunov and Serkov, 2019, Theorem 1) on the basis of Proposition 1 and the lemma below. It should be noted that, as in the reference mentioned, all the statements in the paper are valid for arbitrary partitions, which do not necessarily have a constant step.

Lemma 1. For any $\xi > 0$, there exists $\varepsilon_* \in (0, 1)$ such that, for any $\varepsilon \in (0, \varepsilon_*]$ and any compact in $L_1(T, \mathbb{R}^q)$ set $V \subset \mathcal{V}$, one can choose $\delta_* > 0$ such that, for any initial state $x_0 \in G_0$ and any partition Δ (3) with the constant step $\delta \le \delta_*$, one can specify the values of the parameters $\alpha > 0$ and $\beta > 0$ such that the following is valid. Let the motions $x(\cdot)$ and $y(\cdot)$ of system (1) be generated from the initial states x_0 and $\widetilde{x}_0$ by realizations $u(\cdot)$, $v(\cdot)$ and $\overline{u}(\cdot)$, $\overline{v}(\cdot)$, respectively. Let the inclusion $v(\cdot) \in V$ hold and relations (5)–(11) and (13) be satisfied. Then

$$\|x(t) - y(t)\| \le \xi,\ t \in T. \quad (14)$$

Proof. The proof follows the scheme from (Gomoyunov and Serkov, 2019, Lemma 1). In the arguments below, and in the proof of Lemma 2 in Sect. 5, we emphasize only the differences that arise due to the presence of the informational and computational errors.

Let $\varkappa > 0$ and $L > 0$ be such that $\|f(t, x, u, v)\| \le \varkappa$ and $\|f(t, x, u, v) - f(t, x', u, v)\| \le L\|x - x'\|$ for any $(t, x), (t', x') \in G$, $u \in P$, and $v \in Q$. Let us denote by μ_t, μ_u and μ_v the moduli of continuity of the function $f = f(t, x, u, v)$, $(t, x) \in G$, $u \in P$, $v \in Q$, with respect to t, u and v, and put $\psi(\delta) = \mu_t(\delta) + L\varkappa\delta$, $\delta > 0$. Let $\xi > 0$ be fixed. Let $\xi_* > 0$ and $\varepsilon_* \in (0, 1)$ be such that

$$\xi_* e^{L(\vartheta - t_0)} \le \xi,\ 2(\vartheta - t_0)(\varepsilon_* \varkappa + \mu_u(\varepsilon_*)) \le \xi_*/3. \quad (15)$$

Let $\varepsilon \in (0, \varepsilon_*]$ and a compact in $L_1(T, \mathbb{R}^q)$ set $V \subset \mathcal{V}$ be fixed. Let us choose $\delta_* > 0$ such that, for any $\delta \in (0, \delta_*]$ and any $v(\cdot) \in V$, the inequality below holds:

$$\begin{aligned} &2(\varkappa\delta + (\vartheta - t_0)(\psi(\delta) + \psi(2\delta))) \\ &+ \frac{2n_\varepsilon}{\varepsilon\delta} \int_{t_0}^{\vartheta} \int_{s-2\delta}^{s+2\delta} \mu_v(\|v(s) - v(\tau)\|)\, d\tau\, ds \le \xi_*/3. \end{aligned}$$

Let a partition Δ with the constant step $\delta \le \delta_*$ be fixed. Let us choose $\alpha > 0$ and $\beta > 0$ from the condition

$$(1 + 2(\vartheta - t_0)(L + 2n_\varepsilon/(\varepsilon\delta))\alpha + (\vartheta - t_0)\beta \le \xi_*/3.$$

We will show that the assertion of the lemma holds for the chosen parameters.

According to (7), for $t \in T$, we have

$$\begin{aligned} &\|x(t) - y(t)\| \\ &\le \alpha + \int_{t_0}^{t} \|f(s, x(s), u(s), v(s)) - f(s, x(s), \overline{u}(s), v(s))\|\, ds \\ &+ \int_{t_0}^{t} \|f(s, x(s), \overline{u}(s), v(s)) - f(s, x(s), \overline{u}(s), \overline{v}(s))\|\, ds \\ &+ \int_{t_0}^{t} \|f(s, x(s), \overline{u}(s), \overline{v}(s)) - f(s, y(s), \overline{u}(s), \overline{v}(s))\|\, ds \\ &= \alpha + I_1 + I_2 + I_3. \end{aligned}$$

We derive $I_1 \le 2\varkappa\varepsilon(\vartheta - t_0)$ and $I_3 \le L \int_{t_0}^{t} \|x(s) - y(s)\|\, ds$. Let us estimate I_2. For $s \in [\tau_0, \tau_1)$, we obtain

$$\|f(s, x(s), \overline{u}(s), v(s)) - f(s, x(s), \overline{u}(s), \overline{v}(s))\| \le 2\varkappa. \quad (16)$$

Now, let $i \in 1..(n_\Delta - 1)$ and $s \in [\tau_i, \tau_{i+1})$. According to (5), there exists $j \in 1..n_\varepsilon$ such that

$$\begin{aligned} &\|f(s, x(s), \overline{u}(s), v(s)) - f(s, x(s), \overline{u}(s), \overline{v}(s))\| \\ &\le \|f(s, x(s), u^\varepsilon_j, v(s)) - f(s, x(s), u^\varepsilon_j, \overline{v}(s))\| + 2\mu_u(\varepsilon). \end{aligned}$$

Further, we derive

$$\begin{aligned} &\|f(s, x(s), u^\varepsilon_j, v(s)) - f(s, x(s), u^\varepsilon_j, \overline{v}(s))\| \\ &\le \|f(s, x(s), u^\varepsilon_j, v(s)) - \widetilde{d}_{ij}\| + \|\widetilde{d}_{ij} - f(s, x(s), u^\varepsilon_j, \overline{v}(s))\|. \end{aligned}$$

For the first term, due to (6), (9) and (11), we obtain

$$\begin{aligned} &\max_{j \in 1..n_\varepsilon} \|f(s, x(s), u^\varepsilon_j, v(s)) - \widetilde{d}_{ij}\| \\ &\le \max_{j \in 1..n_\varepsilon} \|f(s, x(s), u^\varepsilon_j, v(s)) - d_{ij}\| + \max_{j \in 1..n_\varepsilon} \|d_{ij} - \widetilde{d}_{ij}\| \\ &\le \psi(2\delta) + \frac{n_\varepsilon}{\varepsilon\delta} \int_{s-2\delta}^{s+2\delta} \mu_v(\|v(s) - v(\tau)\|)\, d\tau\, ds + \frac{2\alpha n_\varepsilon}{\varepsilon\delta}, \end{aligned}$$

where we denote $d_{ij} = (x(\tau'_{ij}) - x(\tau'_{i(j-1)}))/(\tau'_{ij} - \tau'_{i(j-1)})$. For the second term, by the choice (10) of $\overline{v}(s) = \overline{v}_i$, and since, due to (9), the inequality

$$\|f(\tau_i, \widetilde{x}(\tau_i), u, v) - f(s, x(s), u, v)\| \le L\alpha + \psi(\delta) \quad (17)$$

is valid for any $u \in P$ and $v \in Q$, we deduce

$$\begin{aligned} &\|\widetilde{d}_{ij} - f(s, x(s), u^\varepsilon_j, \overline{v}_i)\| \\ &\le \max_{j \in 1..n_\varepsilon} \|\widetilde{d}_{ij} - f(\tau_i, \widetilde{x}(\tau_i), u^\varepsilon_j, \overline{v}_i)\| + L\alpha + \psi(\delta) \end{aligned}$$

$$\leq \max_{j\in 1..n_\varepsilon} \|\widetilde{d}_{ij} - f(\tau_i, \widetilde{x}(\tau_i), u^\varepsilon_j, v(s))\| + \beta + L\alpha + \psi(\delta)$$
$$\leq \max_{j\in 1..n_\varepsilon} \|\widetilde{d}_{ij} - f(s, x(s), u^\varepsilon_j, v(s))\| + \beta + 2L\alpha + 2\psi(\delta).$$

Thus, for I_2, according to the choice of δ_*, we have

$$I_2 \leq \xi_*/3 + (\vartheta - t_0)(2\mu_u(\varepsilon) + \beta + 2L\alpha + 4\alpha n_\varepsilon/(\varepsilon\delta)).$$

Summarizing, due to the choice of ε^*, α and β, we obtain

$$\|x(t) - y(t)\| \leq \xi_* + L\int_{t_0}^{t} \|x(s) - y(s)\|\,\mathrm{d}s,\ t \in T, \quad (18)$$

wherefrom, applying Bellman–Gronwall lemma and taking into account the choice of ξ_*, we derive inequality (14). ■

5. PARTICULAR CASE

In the numerical realization of the above control procedure, the rapid growth when $\varepsilon \downarrow 0$ of the complexity of the disturbance reconstruction problem (10) can cause difficulties. In this section, the case is considered when the function f from (1) satisfies the additional assumption below (see Serkov (2013); Gomoyunov and Serkov (2017)).

Assumption 1. For any $(t, x) \in G$ and $v, v' \in Q$, if the equality $f(t, x, u, v) = f(t, x, u, v')$ holds for some $u = u' \in P$, then this equality holds for any $u \in P$.

In this case, to reconstruct the disturbance, it is sufficient to use any single value of the control instead of the series of "test" controls u^ε_j, $j \in 1..n_\varepsilon$. In particular, if we simply use the control value from the previous step, we obtain the following control procedure.

Let $x_0 \in G_0$, $\varepsilon \in (0, 1)$ and Δ be a partition (3) with the constant step $\delta > 0$. Let $\alpha > 0$ and $\beta > 0$ be fixed. Let an initial state of the guide $\widetilde{x}_0$ satisfies (7). We define a piecewise constant control realization

$$u(t) = u_i \in P,\ t \in [\tau_i, \tau_{i+1}),\ i \in 0..(n_\Delta - 1), \quad (19)$$

in the original system and realizations $\overline{u}(\cdot)$ and $\overline{v}(\cdot)$ of the form (8) in the guide according to the following rule.

Let $i \in 0..(n_\Delta - 1)$. If $i = 0$, we choose $\overline{v}_0 \in Q$ arbitrarily. For $i > 0$, we assume that the approximate values $\widetilde{x}(\tau_{i-1})$ and $\widetilde{x}(\tau_i)$ of the state vector are known such that

$$(\tau_{i-1}, \widetilde{x}(\tau_{i-1})), (\tau_i, \widetilde{x}(\tau_i)) \in G,$$
$$\|x(\tau_{i-1}) - \widetilde{x}(\tau_{i-1})\| \leq \alpha,\ \|x(\tau_i) - \widetilde{x}(\tau_i)\| \leq \alpha, \quad (20)$$

and "reconstruct" the disturbance on the interval $[\tau_{i-1}, \tau_i)$ by choosing $\overline{v}_i \in Q$ from the condition

$$\|\widetilde{d}_{i1} - f(\tau_i, \widetilde{x}(\tau_i), \overline{u}_{i-1}, \overline{v}_i)\|$$
$$\leq \min_{v\in Q} \|\widetilde{d}_{i1} - f(\tau_i, \widetilde{x}(\tau_i), \overline{u}_{i-1}, v)\| + \beta, \quad (21)$$

where

$$\widetilde{d}_{i1} = (\widetilde{x}(\tau_i) - \widetilde{x}(\tau_{i-1}))/(\tau_i - \tau_{i-1}). \quad (22)$$

Further, using the optimal counter–strategy in the guide, we define $\overline{u}_i$ according to (12) and set

$$u_i = \overline{u}_i. \quad (23)$$

Theorem 2. Let Assumption 1 be satisfied and $\mathbb{V}$ be a compact functional constraint on the disturbance. Then, for any $\zeta > 0$, there exists $\varepsilon^* \in (0, 1)$ such that, for any $\varepsilon \in (0, \varepsilon^*]$ and any $V \in \mathbb{V}$, one can choose $\delta^* > 0$ such that, for any initial state $x_0 \in G_0$ and any partition Δ (3) with the constant step $\delta \leq \delta^*$, one can specify the values of the parameters $\alpha > 0$ and $\beta > 0$ such that the control procedure with a guide (7), (8), (12) and (19)–(23) provides inequality (4) for any $v(\cdot) \in V$.

The theorem follows from Proposition 1 and the lemma below (see (Gomoyunov and Serkov, 2019, Theorem 1)).

Lemma 2. Let Assumption 1 be satisfied. Then, for any $\xi > 0$ and any compact in $L_1(T, \mathbb{R}^q)$ set $V \subset \mathcal{V}$, one can choose $\delta_* > 0$ such that, for any initial state $x_0 \in G_0$ and any partition Δ (3) with the constant step $\delta \leq \delta_*$, one can specify the values of the parameters $\alpha > 0$ and $\beta > 0$ such that the following is valid. Let the motions $x(\cdot)$ and $y(\cdot)$ of system (1) be generated from the initial states x_0 and $\widetilde{x}_0$ by realizations $u(\cdot)$, $v(\cdot)$ and $\overline{u}(\cdot)$, $\overline{v}(\cdot)$, respectively. Let the inclusion $v(\cdot) \in V$ hold and relations (7), (8) and (19)–(23) be satisfied. Then inequality (14) is valid.

Proof. We follow the scheme from (Gomoyunov and Serkov, 2019, Lemma 2).

For $\delta > 0$, let us denote

$$\mu_{uv}(\delta) = \max\Big\{\|f(t, x, u, v) - f(t, x, u, v')\| : (t, x) \in G,$$
$$u, u' \in P,\ v, v' \in Q,\ \|f(t, x, u', v) - f(t, x, u', v')\| \leq \delta\Big\}.$$

Note that, $\lim_{\delta\downarrow 0} \mu_{uv}(\delta) = 0$ due to Assumption 1, and, for any $(t, x) \in G$, $u, u' \in P$, $v, v' \in Q$,

$$\|f(t, x, u, v) - f(t, x, u, v')\|$$
$$\leq \mu_{uv}\big(\|f(t, x, u', v) - f(t, x, u', v')\|\big). \quad (24)$$

Let $\xi > 0$ and a compact in $L_1(T, \mathbb{R}^q)$ set $V \subset \mathcal{V}$ be fixed. Let $\xi_* > 0$ satisfy the first inequality in (15). Basing on a suitable modification of (Gomoyunov and Serkov, 2017, Assertion 2), one can show that there exist $\delta_* > 0$ and $\eta_* > 0$ such that, for any $\delta \in (0, \delta_*]$ and $v(\cdot) \in V$, the inequality below is valid:

$$2\varkappa\delta + \int_{t_0}^{\vartheta} \mu_{uv}\bigg(\eta_* + 2\psi(2\delta) + 2\psi(\delta)$$
$$+\frac{2}{\delta}\int_{s-2\delta}^{s+2\delta} \mu_v(\|v(s) - v(\tau)\|)\,\mathrm{d}\tau\bigg)\,\mathrm{d}s \leq \xi_*/2.$$

Let a partition Δ with the constant step $\delta \leq \delta_*$ be fixed. Let us choose $\alpha > 0$ and $\beta > 0$ from the conditions

$$\alpha \leq \xi_*/2,\ 2\alpha/\delta + 2L\alpha + \beta \leq \eta_*.$$

We will show that the assertion of the lemma holds for the chosen parameters.

According to (7) and (23), for $t \in T$, we have

$$\|x(t) - y(t)\|$$
$$\leq \alpha + \int_{t_0}^{t} \|f(s, x(s), \overline{u}(s), v(s)) - f(s, x(s), \overline{u}(s), \overline{v}(s))\|\,\mathrm{d}s$$
$$+\int_{t_0}^{t} \|f(s, x(s), \overline{u}(s), \overline{v}(s)) - f(s, y(s), \overline{u}(s), \overline{v}(s))\|\,\mathrm{d}s$$
$$= \alpha + I_1 + I_2.$$

For I_2, we derive $I_2 \leq \int_{t_0}^{t} L\|x(s) - y(s)\|\,\mathrm{d}s$. Let us estimate I_1. For $s \in [\tau_0, \tau_1)$, inequality (16) holds. Now, let $i \in 1..(n_\Delta - 1)$ and $s \in [\tau_i, \tau_{i+1})$. We have

$$\|f(s, x(s), \overline{u}_{i-1}, v(s)) - f(s, x(s), \overline{u}_{i-1}, \overline{v}(s))\|$$
$$\leq \|f(s, x(s), \overline{u}_{i-1}, v(s)) - \widetilde{d}_{i1}\|$$
$$+\|\widetilde{d}_{i1} - f(s, x(s), \overline{u}_{i-1}, \overline{v}(s))\|.$$

For the first term, due to (20) and (22), we obtain

$$\begin{aligned}&\|f(s,x(s),\overline{u}_{i-1},v(s))-\widetilde{d}_{i1}\|\\ &\le\|f(s,x(s),\overline{u}_{i-1},v(s))-d_{i1}\|+\|d_{i1}-\widetilde{d}_{i1}\|\\ &\le\psi(2\delta)+\frac{1}{\delta}\int_{s-2\delta}^{s+2\delta}\mu_v(\|v(s)-v(\tau)\|)\,\mathrm{d}\tau+\frac{2\alpha}{\delta},\end{aligned}$$

where we denote $d_{i1}=(x(\tau_i)-x(\tau_{i-1}))/(\tau_i-\tau_{i-1})$. For the second term, taking into account (17) and the choice (21) of $\overline{v}(s)=\overline{v}_i$, we deduce

$$\begin{aligned}&\|\widetilde{d}_{i1}-f(s,x(s),\overline{u}_{i-1},\overline{v}_i)\|\\ &\le\|\widetilde{d}_{i1}-f(\tau_i,\widetilde{x}(\tau_i),\overline{u}_{i-1},\overline{v}_i)\|+L\alpha+\psi(\delta)\\ &\le\|\widetilde{d}_{i1}-f(\tau_i,\widetilde{x}(\tau_i),\overline{u}_{i-1},v(s))\|+\beta+L\alpha+\psi(\delta)\\ &\le\|\widetilde{d}_{i1}-f(s,x(s),\overline{u}_{i-1},v(s))\|+\beta+2L\alpha+2\psi(\delta).\end{aligned}$$

Hence, according to the choice of α and β, we have

$$\begin{aligned}&\|f(s,x(s),\overline{u}_{i-1},v(s))-f(s,x(s),\overline{u}_{i-1},\overline{v}(s))\|\\ &\le 2\psi(2\delta)+2\psi(\delta)+\frac{2}{\delta}\int_{s-2\delta}^{s+2\delta}\mu_v(\|v(s)-v(\tau)\|)\,\mathrm{d}\tau+\eta_*.\end{aligned}$$

Then, from (24), by the choice of δ_* and η_*, it follows that $I_1\le\xi_*/2$. Summarizing, due to the choice of α, we obtain (18), wherefrom, by the choice of ξ_*, we derive (14). ■

6. EXAMPLES

Let us illustrate the obtained results by two examples. Both examples belong to the so-called linear-convex case, so we apply the upper convex hulls method (see, e.g., Gomoyunov and Kornev (2016); Kornev (2012)) for calculating $\Gamma^0(x_0)$ and constructing the optimal counter-strategy U^0. As a set V, we choose a set of all functions from T to Q that are piecewise constant on the partition of T with the constant step 0.05. Note that, this set is compact in $L_1(T,\mathbb{R}^q)$. In the numerical simulation, disturbance realizations $v(\cdot)\in V$ are formed on the basis of the optimal counter-strategy of the disturbance, which is also constructed by the upper convex hulls method.

For the control procedures, we choose $\varepsilon=0.01$, $\delta=0.002$. The approximate values $\widetilde{x}(t)$ are generated by a pseudo-random mechanism. As the solution of approximate inequality (10), we use a solution of this inequality with $\beta=0$ and $Q=Q_m$, where Q_m is the m times less accurate discrete approximation of the initial set Q.

The results of the simulation in the examples are presented in Tables 1, 2, where, in every cell, the upper number is the realized value of the quality index and the lower one is the maximal distance between the motions of the system $x(\cdot)$ and the guide $y(\cdot)$. For some parameters (which are in bold), these motions are shown in Figures 1–4. The target points, used in the quality index, are marked by red labels.

Example 1. (General Case). Let us consider a guarantee optimization problem described by the dynamical system

$$\begin{cases}\dot{x}_1(t)=x_2(t),\\ \dot{x}_2(t)=-te^{0.2t}x_1(t)-0.002e^{0.2t}x_2(t)\\ \qquad+u_1(t)\cos v_1(t)-u_2(t)\sin v_1(t)+e^{0.2t}v_2(t),\\ \dot{x}_3(t)=x_4(t),\\ \dot{x}_4(t)=-te^{0.2t}x_3(t)-0.002e^{0.2t}x_4(t)\\ \qquad+u_1(t)\sin v_1(t)+u_2(t)\cos v_1(t)+e^{0.2t}v_3(t),\end{cases}$$

$$t\in[0,4],\ x(t)=(x_1(t),x_2(t),x_3(t),x_4(t))\in\mathbb{R}^4,$$
$$(u_1(t),u_2(t))\in P=\{(1,0),(0,1),(-1,0),(0,-1)\},$$
$$v_1(t)\in\{-\pi/4,\pi/4\},\ v_2^2(t)+v_3^2(t)\le 1,$$

the initial condition $x(0)=(1,-1,1,1)$ and the quality index

$$\gamma=\left((x_1(2)+0.5)^2+(x_3(3)+2)^2+x_1^2(4)+(x_3(4)-2)^2\right)^{\frac{1}{2}}.$$

For this problem, the obtained value of the optimal guaranteed result is $\Gamma^0\approx 1.49$. In the simulation, we use the control procedure described in Sect. 4, where we choose $n_\varepsilon=4$ and take the whole set P as its ε-net.

Table 1. The results of the simulation in Ex. 1

$\alpha\downarrow$	$m\rightarrow$	1	2	4	8
10^{-4}	γ	2.5346	2.6059	2.6125	2.7831
	$\|x(\cdot)-y(\cdot)\|$	4.1068	4.4605	4.3375	4.8385
10^{-5}	γ	1.6286	1.6187	1.6497	**1.6697**
	$\|x(\cdot)-y(\cdot)\|$	1.5901	1.5514	1.7135	**1.8525**
10^{-6}	γ	1.2693	**1.2681**	1.3443	1.3597
	$\|x(\cdot)-y(\cdot)\|$	0.1980	**0.1847**	0.3748	0.6342
10^{-7}	γ	1.2660	1.2639	1.3418	1.3597
	$\|x(\cdot)-y(\cdot)\|$	0.1213	0.1133	0.3523	0.6342

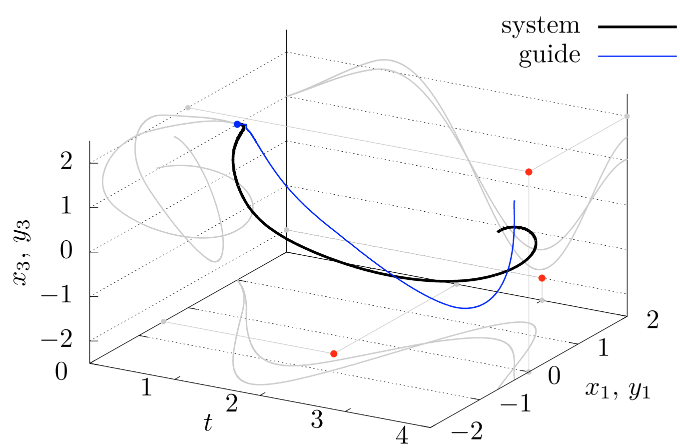

Fig. 1. Ex. 1: the case $(\alpha,m)=(10^{-5},8)$.

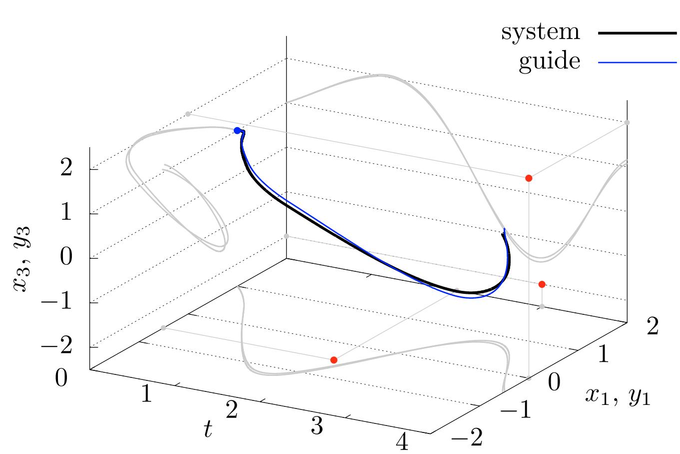

Fig. 2. Ex. 1: the case $(\alpha,m)=(10^{-6},2)$.

Example 2. (Particular Case). Let us consider a guarantee optimization problem described by the dynamical system

$$\begin{cases} \dot{x}_1(t) = u_1(t)(v_1(t) + v_2(t)), \\ \dot{x}_2(t) = u_2(t)v_1(t)v_2(t), \end{cases} \tag{25}$$

$$t \in [0,2],\ x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2,$$

$$0.5 \le |u_i(t)| \le 1.5,\ i = 1,2, \quad 1 \le v_1^2(t) + v_2^2(t) \le 4.$$

the initial condition $x(0) = (0,0)$, and the quality index

$$\gamma = \big((x_1(1) - 2)^2 + (x_2(1) - 1)^2 + x_1^2(2) + (x_2(2) + 2)^2\big)^{\frac{1}{2}}.$$

For this problem, the obtained value of the optimal guaranteed result is $\Gamma^0 \approx 2.8759$. One can verify that system (25) satisfies Assumption 1. Therefore, in the simulation, we use the simplified control procedure from Sect. 5.

Table 2. The results of the simulation in Ex. 2

α ↓	m →	1	2	4	8
10^{-1}	γ	3.1598	3.4616	3.1883	3.4552
	$\|x(\cdot) - y(\cdot)\|$	1.9474	2.2069	1.9277	2.1618
10^{-2}	γ	2.3277	2.2974	2.2989	**2.3585**
	$\|x(\cdot) - y(\cdot)\|$	1.1617	1.2078	1.1216	**1.2237**
10^{-3}	γ	1.4638	**1.5519**	1.5031	1.5365
	$\|x(\cdot) - y(\cdot)\|$	0.1008	**0.1653**	0.1658	0.3376
10^{-4}	γ	1.5384	1.5325	1.4862	1.5092
	$\|x(\cdot) - y(\cdot)\|$	0.2417	0.1433	0.2876	0.3692

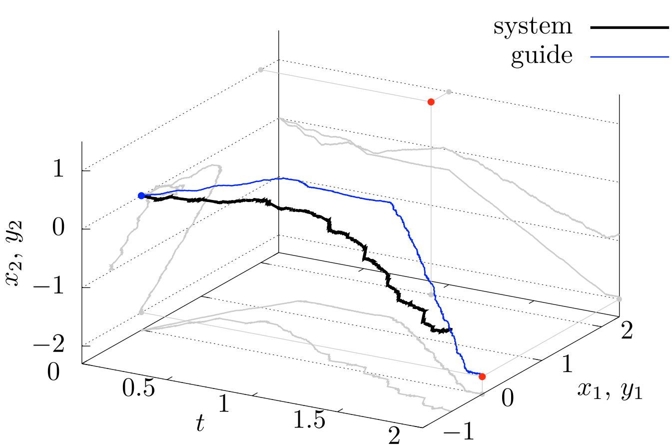

Fig. 3. Ex. 2: the case $(\alpha, m) = (10^{-2}, 8)$.

As expected, the simplified control procedure, when it can be applied, gives more stable results. Namely, the ratio of the threshold values of the parameter α for the satisfactory work of the control procedures in Examples 1 and 2 equals to $10^{-3}/10^{-6} = 10^3$ (see Tables 1 and 2). Note that this ratio has the same order as the value $n_\varepsilon/\varepsilon \approx 0.4 \times 10^3$ (see relations (6), (11) and (22)).

7. CONCLUSION

In the paper, we have studied the guarantee optimization problem under a compact functional constraint on the disturbance. We have considered the control procedure with a guide that ensures the optimal guaranteed result and its simplified variant for the particular case. We have proved that these procedures are stable with respect to computational and informational errors. The examples have been considered that illustrate the obtained theoretical results.

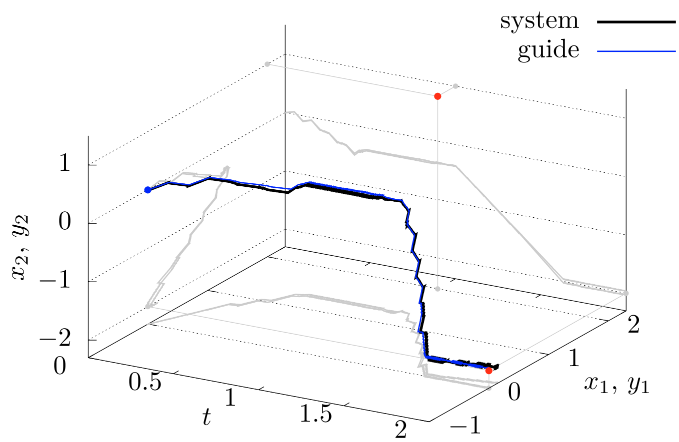

Fig. 4. Ex. 2: the case $(\alpha, m) = (10^{-3}, 2)$.

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